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Coulomb Green's Function in Lobachevsky Space

Yu. Kurochkin* and V. Otchik
B. I. Stepanov Institute of Physics of NAS of Belarus,
68 Nezavisimosti Ave., 220072 Minsk, BELARUS
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The Green's function of the Schrödinger equation with the Coulomb potential in the space of constant negative curvature is constructed in the form of an eigenfunction expansion. The closed form expression for this function is also presented.

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The quantum mechanical Kepler–Coulomb problem on a three-dimensional sphere was first solved by Schrödinger [1]. Since then, this problem in spaces of constant curvature was analyzed by many authors from different viewpoints (see e.g. [2–7] and references therein).

Here we consider Green's function for the Schrödinger equation with the Coulomb potential in the three-dimensional space of constant negative curvature (Lobachevsky space).

We use the embedding of the Lobachevsky space in a four-dimensional pseudoeuclidean space with Cartesian coordinates x_μ, $\mu = 1, 2, 3, 4$:

$$x_\mu x_\mu = \mathbf{x}^2 + x_4^2 = \mathbf{x}^2 - x_0^2 = -\rho^2,$$
$$\mathbf{x} = \{x_1,\, x_2,\, x_3\},\ x_4 = i x_0.$$

Commonly used spherical coordinates of the Lobachevsky space β, θ, ϕ are defined by relations

$$\begin{aligned} &x_0 = \rho\cosh\beta,\ x_1 = \rho\sinh\beta\sin\theta\cos\phi,\\ &x_2 = \rho\sinh\beta\sin\theta\sin\phi,\\ &x_3 = \rho\sinh\beta\cos\theta,\\ &0 \le \beta < \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi. \end{aligned} \tag{1}$$

Line and volume elements in these coordinates are

$$\begin{aligned} &ds^2 = \rho^2(d\beta^2 + \sinh^2\beta(d\theta^2 + \sin^2\theta d\phi^2)),\\ &dV = \rho^3\sinh^2\beta\sin\theta d\beta d\theta d\phi. \end{aligned} \tag{2}$$

Schrödinger equation for a particle in an external field with potential U in the Lobachevsky space is written in the form ($\hbar = m = 1$)

$$\triangle\Psi + 2(E - U)\Psi = 0 \tag{3}$$

where $\triangle$ is the Laplace operator in the Lobachevsky space,

$$\triangle = -\frac{1}{2\rho^2}M_{\mu\nu}M_{\mu\nu},\, M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu.$$

Let $\overset{1}{x}_\mu, \overset{2}{x}_\mu$ denote the Cartesian coordinates of two different points in the Lobachevsky space. The spherical coordinates of these points are denoted $\beta_i, \theta_i, \phi_i$, $i = 1, 2$. Then the Green's function satisfies an inhomogeneous equation

$$[\triangle + 2(E - U)]G(\overset{2}{x}, \overset{1}{x}, E) = \delta^3(\overset{2}{x}, \overset{1}{x}) \tag{4}$$

where the delta function is defined by relations

$$\delta^3(x, x') = \frac{\delta(\beta-\beta')\delta(\theta-\theta')\delta(\phi-\phi')}{\rho^3\sinh^2\beta\sin\theta},$$
$$\int \delta^3(x, x') f(x') dV = f(x).$$

Solutions of equation (4) can be written in the form of eigenfunction expansions in terms of simultaneous eigenfunctions of the Hamiltonian and angular momentum operator.

The Coulomb potential in the Lobachevsky space we write as

$$U = -\frac{\alpha}{\rho}\frac{x_0}{|\mathbf{x}|} = -\frac{\alpha\coth\beta}{\rho} \tag{5}$$

*E-mail: `yukuroch@dragon.bas-net.by`

where $\alpha > 0$ for an attractive potential. The normalized solutions of the Schrödinger equation (3) with the potential (5) for the continuous and discrete energy values can be written in the form

$$\begin{aligned} \psi_{Elm}(x) &= N_E S_E^1(\beta) Y_l^m(\theta,\phi), \\ \psi_{nlm}(x) &= N_n S_n(\beta) Y_l^m(\theta,\phi) \end{aligned} \tag{6}$$

where $Y_l^m(\theta,\phi)$ are the spherical harmonics, and dependence on β is expressed in terms of hypergeometric functions:

$$\begin{aligned} &S_E^1(\beta) = \sinh^l \beta e^{(ip-l-1)\beta} {}_2F_1(\sigma+l+1, l-\nu+1; 2l+2; 1-e^{-2\beta}) \\ &(p = \sqrt{2E\rho^2 + 2\alpha\rho - 1},\ \nu - \sigma = ip,\ \sigma = \alpha\rho/\nu); \\ &S_n(\beta) = \sinh^l \beta e^{(-\alpha\rho/n+n-l-1)\beta} {}_2F_1(\alpha\rho/n+l+1, l-n+1; 2l+2; 1-e^{-2\beta}). \end{aligned}$$

The normalization coefficients are

$$N_E = \frac{2^{l+1/2}\sqrt{\sinh \pi p}|\Gamma(l+\nu+1)||\Gamma(l+\sigma+1)|}{\pi\sqrt{\rho}(2l+1)!},$$

$$N_n = \frac{2^{l+1}}{(2l+1)!}\sqrt{\left(\frac{\alpha^2}{\rho n^3} - \frac{n}{\rho^3}\right)\frac{(n+l)!\Gamma(\alpha\rho/n+l+1)}{(n-l-1)!\Gamma(\alpha\rho/n-l)}}.$$

The eigenfunction expansion for the Coulomb Green's function in terms of the solutions (6) is

$$\begin{aligned} G(\overset{2}{x},\overset{1}{x},E_0) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \int\limits_{-\frac{\alpha}{\rho}+\frac{1}{2\rho^2}}^{\infty} \frac{\psi_{Elm}(\overset{2}{x})\psi^*_{Elm}(\overset{1}{x})}{E_0 - E} dE \\ + \sum_{n=1}^{[\alpha\rho]}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} \frac{\psi_{nlm}(\overset{2}{x})\psi^*_{nlm}(\overset{1}{x})}{E_0 - E_n}. \end{aligned} \tag{7}$$

In the integral (7), E_0 is taken on the upper side of the real axis ($\mathrm{Im}E_0 > 0$) to provide the diverging wave behaviour of the Green's function (see discussion in [8] for the case of a flat space).

By performing integration in E, we arrive at the characteristic form of Green's function

$$G(\overset{2}{x},\overset{1}{x},E) = -\frac{1}{4\pi}\sum_{l=0}^{\infty} \frac{\Gamma(l+1-\nu)\Gamma(l+\sigma+1)S_E^1(\beta_<)S_E^2(\beta_>)P_l(\cos\theta_{12})}{\rho\Gamma(1-ip)(2l)!} \tag{8}$$

where $S_E^2(\beta) = \sinh^{-l-1}\beta e^{(ip+l)\beta} {}_2F_1(\sigma-l, -\nu-l; 1-ip; e^{-2\beta})$,

$$\beta_> = \begin{cases} \beta_1, \text{ if } \ \beta_1 > \beta_2, \\ \beta_2, \text{ if } \ \beta_2 > \beta_1, \end{cases}$$

and similar for $\beta_<$, $\cos\theta_{12} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2)$.

In papers of Hostler [9, 10], a closed form expression for the nonrelativistic Coulomb Green's function in a flat space has been obtained by summing the partial wave expansion. It is possible to extend Hostler's approach to the case of the Lobachevsky space. We only formulate here a final result. By using notations

$$f_E^1(\beta) = \sinh\beta S_{E0}^1(\beta) = \sinh\beta e^{(ip-1)\beta}{}_2F_1(\sigma+1, 1-\nu; 2; 1-e^{-2\beta}),$$
$$f_E^2(\beta) = \sinh\beta S_{E0}^2(\beta) = e^{ip\beta}{}_2F_1(\sigma, -\nu; 1-ip; e^{-2\beta}).$$

we can write a closed expression for the Coulomb Green's function in the Lobachevsky space:

$$G(\overset{2}{x}, \overset{1}{x}, E) = -\frac{\Gamma(1-\nu)\Gamma(\sigma+1)}{4\pi\rho\sinh\beta_{12}\Gamma(1-ip)}\left(\frac{\partial}{\partial\beta_-} - \frac{\partial}{\partial\beta_+}\right) f_E^1(\beta_-) f_E^2(\beta_+) \tag{9}$$

where

$$\beta_\pm = (\beta_1 + \beta_2 \pm \beta_{12})/2, \cosh\beta_{12}$$
$$= \cosh\beta_1\cosh\beta_2 - \sinh\beta_1\sinh\beta_2\cos\theta_{12}.$$

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